Peculiar Periodicity in Press Patterns; Rectangular Lights Out Kernel Dimensions are Periodic

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June 6, 2024

Abstract

The *Lights Out* puzzle game has been extensively analyzed before, using linear algebra. In this paper we discuss and prove interesting observations about dimensions of null spaces determining the number of solutions. In particular, we find periodic patterns in these dimensions, when the size of the game board as well as the number of colours of the buttons, varies.

Introduction

Lights Out is a handheld electronic puzzle game produced by Tiger Electronics in the 1990s. It consists of a square grid of 25 buttons that also act as lights. Each light has two states: on and off. Pressing a button has the effect of changing the state of the light of the button itself as well as that of each of its four possible direct neighbours to the left, right, above, and below. The object of the game is to turn off all the lights (from some initial configuration) by pressing a series of buttons.

The main goal of this paper is to analyze the number of solutions of the game and certain generalizations. The generalizations concern the size of the board (we consider rectangular boards of any size $r \times c$), and the number of colours: besides the off-state we will allow not just one on-state, but any positive number of different on-states (the different colours). Pressing a button repeatedly will change the state of the button (and its neighbours) in a fixed, cyclic order. We denote the number of states by $n \geq 2$; the case $n = 2$ is that of the original game. The number of solutions we study refers to both the number of initial states of a board that can be solved, and to the number of different solutions for such cases.

By previous work it was clear (as we will explain in Section 2) that the numbers we are looking for are the numbers of elements of certain linear spaces determined by a matrix associated with the game. The results in this paper were inspired by observations we made when inspecting tables like the one shown here as Table 1. The table encodes part of all the information we search for in the case of rectangular boards of up to 15 columns and 32 rows with lights that can either be on or off, as in the original Lights Out game (with 5 columns and 5 rows).

Each entry $d = d(r, c)$ signifies the dimension of a vector space over $\mathbb{F}_2$ (the 2 coming from the number of states, n) containing a number of vectors (2^d) corresponding to the number of different solutions for any solvable initial board configuration of size $r \times c$. Moreover, the number of different initial configurations for an $r \times c$ board that are solvable, equals $2^{r \cdot c - d}$, as we will soon see. Thus, the number $d(5, 5) = 2$ in the table indicates that for the original Lights Out game $2^{5 \cdot 5 - 2} = 2^{23}$ out of the 2^{25} possible initial configurations are solvable, and for each of these there will be $2^2 = 4$ different ways to solve it. Note that zeroes in the table occur precisely when *every configuration of lights on the $r \times c$ board can be turned off, each in a* unique way.

The main new results in this paper concern the regular patterns in this table and similar versions for larger n, where the situation is slightly trickier if n is a composite integer. Our initial observations for Table 1 (extended in both directions) can be summarized as follows.

For $c \geq 0$, consider the column $d_c(r)$ of non-negative integers $d(r, c)$ for boards with a fixed number of columns c and $r = 0, 1, 2, 3, \ldots$ rows. By symmetry, the c-th row coincides with the c-th column, so the properties we list here also apply to the rows. We find:

- **Observation 1** Periodicity of columns:
  For every $c \geq 0$ the sequence d_c is purely periodic.

- **Observation 2** In each column the maximum value is attained:

  $$\forall c \geq 0 \; \exists r \geq 0 : d_c(r) = c.$$

  By r_0 we will denote the *smallest* such r.

- **Observation 3** Columns repeat after $r_0 + 1$ entries:
  The period length ℓ of d_c equals $r_0 + 1$.

$r \setminus c$:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	ℓ
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	3
2	0	1	0	2	0	1	0	2	0	1	0	2	0	1	0	2	4
3	0	0	2	0	0	3	0	0	2	0	0	3	0	0	2	0	6
4	0	0	0	0	4	0	0	0	0	4	0	0	0	0	4	0	5
5	0	1	1	3	0	2	0	4	1	1	0	4	0	1	1	4	24
6	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	9
7	0	0	2	0	0	4	0	0	2	0	0	7	0	0	2	0	12
8	0	1	0	2	0	1	6	2	0	1	0	2	0	7	0	2	28
9	0	0	1	0	4	1	0	0	1	8	0	1	0	0	5	0	30
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	31
11	0	1	2	3	0	4	0	7	2	1	0	6	0	1	2	8	48
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	63
13	0	0	1	0	0	1	0	0	7	0	0	1	0	0	1	0	18
14	0	1	0	2	4	1	0	2	0	5	0	2	0	1	4	2	340
15	0	0	2	0	0	4	0	0	2	0	0	8	0	0	2	0	24
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	0	255
17	0	1	1	3	0	2	6	4	1	1	0	4	0	13	1	4	168
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	513
19	0	0	2	0	4	3	0	0	2	8	0	3	0	0	6	0	60
20	0	1	0	2	0	1	0	2	6	1	0	2	0	1	0	2	2340
21	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	186
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2047
23	0	1	2	3	0	5	0	7	2	1	0	10	0	1	2	15	96
24	0	0	0	0	4	0	0	0	0	4	0	0	0	0	4	0	1025
25	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	126
26	0	1	0	2	0	1	6	2	0	1	0	2	0	7	0	2	2044
27	0	0	2	0	0	3	0	0	8	0	0	3	0	0	2	0	36
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3277
29	0	1	1	3	4	2	0	4	1	9	0	4	0	1	5	4	2040
30	0	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	341
31	0	0	2	0	0	4	0	0	2	0	0	8	0	0	2	0	48
32	0	1	0	2	0	1	0	2	0	1	0	2	0	1	0	2	4092

Table 1: Dimension of kernels, and length of period ℓ

- **Observation 4** Sums of two consecutive kernel dimensions are restricted:

$$\forall c\ \forall r \geq 0: \ d_c(r) + d_c(r+1) \leq c.$$

- **Observation 5** Each period is almost palindromic:
  For all c: $d_c(0), d_c(1), \ldots, d_c(r_0 - 1)$ is a palindrome of length r_0, and $d_c(r_0) = c$ completes the period.

Looking, for example, at the initial part of the column with $c = 5$, which contains an entry for the standard 5×5 lay out, we find:

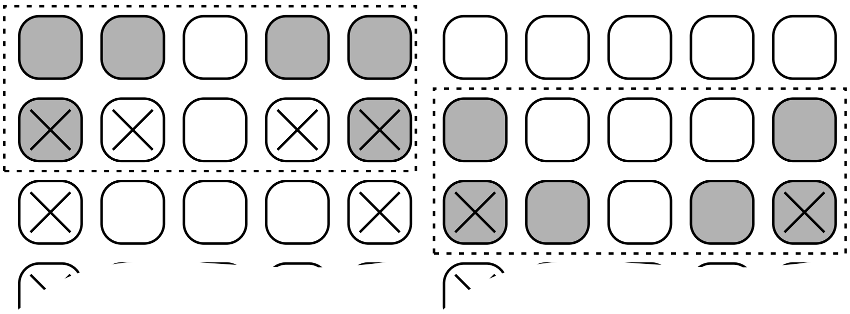
$$0, 1, 1, 3, 0, 2, 0, 4, 1, 1, 0, 4, 0, 1, 1, 4, 0, 2, 0, 3, 1, 1, 0, 5, \ldots$$

and repeats from the beginning after the first '5' occurs, which happens for $r_0 = 23$. The length of the period, $r_0 + 1$, is listed in the final column of the table. The sequence up until the value 5 is palindromic (it reads the same from left to right as from right to left).

The fact that in a column consecutive entries sum to less than the maximal dimension is Observation 4. It implies that the penultimate value in the period (the last value of the palindrome) is always 0, as must be the first.

Prior work

The variants of Lights Out have a considerable history of being studied by mathematicians. In [Fei98], and before that in [Pel87], methods from linear algebra are used to solve Lights Out systematically. The first approach is to number the buttons from 1 to 25 and to identify the state of the board by a row *state vector* s of zeroes and ones (with $s_i = 1$ just for those lights that are lit). This state vector can be interpreted as an element $s \in \mathbb{F}_2^{25}$. Any series of buttons to be pressed will also be coded as the (row) *press vector* $p \in \mathbb{F}_2^{25}$, where $p_i = 1$ if the i-the button is to be pressed, and 0 otherwise. The effect of pressing button i can then be encoded as a row vector $a_i \in \mathbb{F}_2^{25}$, having 1 precisely at the positions of button i and its four (or fewer) direct neighbours: pressing button i then results in adding the *effect vector* a_i to the state vector. Thus the state arrived at from the 0 state (with no lights on) using the press pattern p will be $p \cdot A$, where A is the 25×25 symmetric matrix over $\mathbb{F}_2$ having a_i as its i-th row. Since $v = -v$ for any vector over $\mathbb{F}_2$, it will be clear that the press pattern p needed to turn all lights out from a given initial state vector s can be found as a solution to the vector-matrix equation $p \cdot A = -s = s$.



(a) Window of first two rows (b) Next window

Figure 1: Chasing the lights

Solving this equation by hand is hardly feasible, but the equation does give insight into the solvability question: it turns out that the rank of A is 23, so the kernel has dimension 2. Hence, 2^{23} out of the 2^{25} possible state configurations are solvable, and for each of these there will be $2^2 = 4$ different solutions p.

A more practical solution is given in [Mar01], where a technique known as gathering or *chasing the lights* is introduced. Since the order in which buttons are pressed is of no importance, we may choose to do so row by row of the board, starting from the top, and for ech row from left to right. To turn the lights in a given configuration s off, one might start by turning off the lights in the first row by pressing those buttons in the second row directly below lit lights of the first row.

In Figure 1, for example, some lights are initially lit (indicated by gray). There is only one way to turn off the lit lights of the first row by pressing buttons in the second row, namely, to press each button directly below a lit button (the ones in row 2 marked with $\times$).

Then turn off any remaining lights in the second row by pressing the proper buttons of the third row, and so on. In the end only some buttons in the fifth (or bottom) row will be on. We have chased the lit buttons in the first row down to the last row.

It will be possible to turn off the lit lights in the bottom row if it is possible to create a light pattern t by only pressing buttons on the first row of an empty board, with the property that chasing t results in the same pattern of lit lights on the bottom row.

The task for any person who (like the first author) would like to be able to turn a given light pattern off (when possible) is to memorize the (small table of) results of chasing the independent patterns that can be created on the first row, and to combine those in his or her head to the bottom row pattern obtained by chasing the initial configuration.

In [Lea17] this method of chasing the lights is extended to general rectangular boards. It is shown there, as we will explain further in Section 2, that for analyzing an $r \times c$ Lights Out board one is interested in the upper left $c \times c$ sub-matrix of W^r, where W describes the effect of one step in chasing the lights. As a consequence, both the number of configurations (2^{rs-d}) for which a solution to the Lights Out problem exists and the number of different solutions once it is solvable (2^d), are determined by the dimension d of a linear subspace of some $\mathbb{F}_2$-vector space (the kernel of the submatrix of W^r referred to above); it is this number that is given in Table 1.

In Section 1 we will describe chasing and the matrices involved in detail, because this is the context in which we made the above observations about the 'kernel dimensions for Lights Out on a rectangular board'.

In Section 3 we will prove the observations we made, not just for 2 colours, but any *prime* number of colours. In Section 4 we deal with the complications arising when the number of colours n is a *composite* number.

1 Matrices

We will be exploring *rectangular Lights Out with n colours* $\mathcal{L}(r, c, n)$: the game will consist of an $r \times c$ rectangular layout (so r rows and c columns) of buttons that can each be in one of n different states. These states will often be referred to as *colours*, and will be identified with elements of $\mathbb{Z}/n\mathbb{Z}$. The zero-state of a button means that its light is turned off. Pressing any of the buttons will change the state of the button itself as well as that of any of its neighbouring buttons by adding 1 modulo n. Neighbouring buttons are buttons immediately next to, above it or below it; so there are at most 4 of them (and fewer on the edges). The standard Lights Out game described in the Introduction corresponds to $\mathcal{L}(5, 5, 2)$.

Remark There are two other generalization we like to mention that will not be dicussed further here. In some variants of the game the effect of pressing some button is different: the states of adjacent buttons may be affected by adding other values, or the neighbouring relation may be given by a more general graph than this rectangular one. In another variant of the game one is only allowed to press buttons that are lit, i.e., that are in a state different from zero. The latter requirement drastically alters the game, as the order in which buttons are pressed will be of importance then! $\triangleleft$

An instance of the $\mathcal{L}(r, c, n)$ Lights Out puzzle now consists of an $r \times c$ matrix L, the light pattern, with entries from $\mathbb{Z}/n\mathbb{Z}$, and the goal is to reach the state where all lights are off (the zero matrix), by a sequence of button presses.

The effect on L of pressing the single button at position i, j is given by

$$\pi_{i,j} : L \mapsto L + P_{i,j},$$

for $1 \leq i \leq r$ and $1 \leq j \leq c$, and is called a *basic press*. Here the matrix $P_{i,j}$ has the same size as L, and is defined by

$$P_{i,j}(u, v) := \begin{cases} 1 & : \text{ if } \mathrm{d}\left((i, j), (u, v)\right) \leq 1 \\ 0 & : \text{ otherwise,} \end{cases}$$

where d is the *Manhattan distance* between positions (i, j) and (u, v). Put differently, when pushing the button at position (i, j) the effect will be to add matrix $P_{i,j}$ to L, where $P_{i,j}$ is the $r \times c$ matrix with $1 \in \mathbb{Z}/n\mathbb{Z}$ at position i, j and its immediate neighbours.

A *press pattern* could be thought of as a succession of basic presses; however, it will be clear that the order in which the basic presses are executed in the press pattern is irrelevant, and since repeating a particular basic press n times will have no effect, we may identify a press pattern also by an element $\Pi \in \mathcal{M}_{r,c}(\mathbb{Z}/n\mathbb{Z})$: matrix element $\Pi_{i,j} \in \mathbb{Z}/n\mathbb{Z}$ simply indicates how often the button at position i, j will be pressed. The *effect* of Π on the light pattern L will be:

$$\Pi : L \mapsto L + \sum_{j=1}^{r} \sum_{i=1}^{c} \Pi_{i,j} P_{i,j}.$$

We will call $E = E(\Pi) = \sum_{j=1}^{r} \sum_{i=1}^{c} \Pi_{i,j} P_{i,j}$ the *effect-matrix* of press pattern Π, and thus $\Pi(L) = L + E$. Note that $\Pi_{i,j}$ is a scalar element of $\mathbb{Z}/n\mathbb{Z}$, while $P_{i,j}$ is a matrix over $\mathbb{Z}/n\mathbb{Z}$, and thus $\Pi_{i,j} P_{i,j}$, as well as E, is again an $r \times c$ matrix.

Note that we now have three different interpretations of elements of $\mathcal{M}_{r,c}(\mathbb{Z}/n\mathbb{Z})$: a light pattern L gives the states (colours) of all buttons in the display; a press pattern Π indicates which buttons will be pressed (and how often), and an effect matrix E records the change to L of applying Π.

The composition of two press patterns corresponds to the sum of the matrices $\Pi + \Pi'$, and this makes the set of all press patterns $\mathcal{P}$ into an additive group, in which the identity element corresponds to the zero matrix. It is also clear from the definition that $E(\Pi + \Pi') = E(\Pi) + E(\Pi')$.

Chasing

We will now more formally describe the operation of *chasing* a light pattern L on any rectangular board, with any number of colours. It consists of two steps.

In the first step a press pattern Π is found, with corresponding effect matrix E, such that applying Π to L results in a light pattern chase(L) that has the property that $\text{chase}(L)_{i,j} = 0$ for entries with $i < r$; in other words, only entries on the bottom row of $L + E$ can be non-zero.

For the second step L, a press pattern solution S to some matrix equation is needed, which has the effect that remaining lit lights on the bottom row are also turned off.

Before giving the details of the two steps, we determine the effect of any row of button presses $(p_1, p_2, \ldots, p_c)$. Pressing the first button p_1 times adds p_1 to both the first and the second light in that row (hence to the row of E), pressing the second button p_2 times adds p_2 to the first, second and third light, and so on. In summary, the effect on a row of applying the press pattern $(p_1, p_2, \ldots, p_c)$ to that row is given by the matrix multiplication

$$(p_1, p_2, \ldots, p_c) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{pmatrix}$$

where we will call the $c \times c$ matrix on the right E_c: it has ones only on, directly above, and directly below the main diagonal. By using matrix multiplication on the right, the effect of the row of button presses on the left show the effect on that row as a row vector again. Note that the effect of the press vector on the rows immediately above and below is that of simply adding $(p_1, p_2, \ldots, p_c)$ to that row.

The press pattern matrix Π for the first step is constructed row-by-row: the first row will be zero. The second row of Π is chosen in such a way that the lights on the first row of L will all be turned off; that is, $\Pi_{2,j} = n - L_{1,j}$ for $1 \le j \le c$, meaning that we press the button below any button in the first row exactly so many times that the button in the first row will be turned off. The first row of the effect matrix E will thus be $(n - L_{1,1}, n - L_{1,2}, \ldots, n - L_{1,c})$, as required. Now the complications start, as the second row of the press pattern will also affect the second and third rows of E. The third row will become a copy of the first row, but the second row of E will become the result of $(n - L_{1,1}, n - L_{1,2}, \ldots, n - L_{1,c}) \cdot E_c$, as shown above. Once we have changed the second and third rows of E, we replace L by $L + E$, a new light pattern where all lights in the first row are turned off. Next we determine the third row of Π, in such a way that that also the lights in the second row of the new light pattern will be turned off, adapt the effect matrix E accordingly (rows 2, 3, and 4 will be modified) and change the light pattern again. This is repeated, until by changing the bottom row of Π also the penultimate row of lights in L is turned off completely. The resulting light pattern $\text{chase}(L)$ is thus uniquely determined, and it is completely described by the bottom row $b = (b_1, b_2, \cdots, b_c)$, as all other entries are zero.

For the second step of solving a given Lights Out problem L, a press pattern solution S to the matrix equation $S \cdot E_c = -\text{chase}(L)$ is needed: the effect of first applying the press pattern Π found above and then the press pattern S will turn all the lights off.

Two questions clearly remain: does such solution S exist, and how to find it in case the answer is affirmative?

From a theoretical point of view both questions are answered simultaneously by observing that any light pattern for which only bottom row lights are turned on can be obtained from a board where all lights are initially turned off by just chasing a light pattern created from some press pattern involving only buttons on the first row. Since all operations involved are linear, the only bottom row patterns that can be obtained are the $\mathbb{Z}/n\mathbb{Z}$-linear combinations of the bottom rows $a^{(1)}, a^{(2)}, \ldots, a^{(c)}$, where $a^{(i)}$ is the result of chasing the light pattern that is the effect of only pressing the i-th button on the first row once. A solution S only exists when when $-\text{chase}(L) = -(b_1, b_2, \ldots, b_c)$ is a linear combination of the vectors $a^{(i)}$:

$$-(b_1, b_2, \ldots, b_c) = \lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \cdots + \lambda_c a^{(c)}.$$

For the practical puzzler, the solution can be found if the vectors $a^{(i)}$ are memorized, the coefficients λ_i can be determined by a calculation from the top of one's head, and then the solution is found by chasing the result of applying the press pattern $(\lambda_1, \lambda_2, \ldots, \lambda_c)$ to the first row.

On the other hand, this also implies that the only press patterns that do not change the state of the board, are those that combine a press pattern on the first row by a press pattern on all subsequent rows that correspond to chasing the lights to a bottom row without any lights on.

2 Periodicity

In this section we reformulate the second stage of the chasing operation in terms of matrix multiplication. This will enable us to prove our observations in the Introduction.

Using the square $c \times c$ matrix E_c from the previous section, we define the $2c \times 2c$ matrix W_c with entries $0, 1, -1$ from $\mathbb{Z}/n\mathbb{Z}$ by

$$W_c := \begin{pmatrix} -E_c & -I_c \\ I_c & O_c \end{pmatrix}$$

where O_c and I_c are the $c \times c$ zero and identity matrix.

Consider the row vector $p \oplus o = (p_1, p_2, \ldots, p_c, 0, 0, \ldots, 0)$ of length $2c$ and the result $r = (p \oplus o) \cdot W_c$: it will be clear that

$$r = (e_1, e_2, \ldots, e_c, -p_1, -p_2, \ldots, -p_c),$$

by the considerations of the previous section: here, by definition of W_c, $(e_1, e_2, \ldots, e_c)$ is the effect of the press pattern $p = (p_1, p_2, \ldots, p_c)$. Thus, we can interpret the result of multiplying $p \oplus o$ by W_c as $e \oplus -p$: the first half is the effect on the first row of applying the press pattern p, the second half is the effect on the second row.

Suppose we now multiply this result by W_c again:

$$(e \oplus -p) \cdot W_c = (eE_c - pI_c) \oplus (-eI_c - pO_c) = (eE_c - p) \oplus -e.$$

The resulting vector also has an obvious interpretation: the first half, eE_c+p, is precisely the effect on the second row of chasing the first row, while the second half, $-e$, is the effect of this on the third row.

Repeating this we find the following result.

Lemma 1 *For $k \geq 1$ and for any vector $(p_1, p_2, \ldots, p_c) \in (\mathbb{Z}/n\mathbb{Z})^c$ it holds that*

$$(p_1, p_2, \ldots, p_c) \oplus (0, 0, \ldots, 0) \cdot W_c^k = e_k \oplus e_{k+1},$$

where e_k is the k-th row of chasing the effect of applying press pattern $(p_1, p_2, \ldots, p_c)$ to the first row of a rectangular Lights Out display of c columns. ◇

Note that we did not specify the number of rows r in the rectangular display: one may think of a display with c columns and an arbitrary number of rows.

The reason we are looking at W_c is that its powers tell us something about the effect of chasing down the lights. In particular, if we have an (r, c, n) Lights Out puzzle, the $c \times c$ upper left sub-matrix of W_c^r describes exactly the process of gathering the lights to the last row.

It is important to relate this result to what we were attempting to achieve by chasing in the previous section. Suppose we have an $r \times c$ rectangular board; we concluded that we could solve a given Lights Out problem L if we could find a press pattern for the first row that when chased to the bottom row would give $-\operatorname{chase}(L) = -b = -(b_1, b_2, \ldots, b_c)$, and that if such a solution exists, the number of different solutions equals the number of press patterns for the first row that would be chased to the zero row at the bottom. In terms of the Lemma this means: a solution *exists* if we can find a vector p of length c such that $(p \oplus o) \cdot W_c^r = (-b) \oplus x$, where x can be any vector in $(\mathbb{Z}/n\mathbb{Z})^c$, and the *number of solutions* equals the number of different vectors p for which $(p \oplus o) \cdot W_c^r = o \oplus y$, with y arbitrary.

This is summarized as follows. Here $T_{c,r}$ is the $c \times c$ top left sub-matrix of the power W_c^r of the $2c \times 2c$ matrix W_c defined above.

Corollary 2 *For any $r \times c$ rectangular board for Lights Out with n colours, the number of solvable initial configurations equals the number of different vectors in the row space of $T_{c,r}$ and each of these admits a number of solutions that equals the number of vectors in the kernel of $T_{c,r}$.* $\diamond$

The first interesting fact we prove is that $W_c \in \mathcal{M}_{r,c}(\mathbb{Z}/n\mathbb{Z})$ is invertible.

Lemma 3 *W_c is invertible for all $c \in \mathbb{N}$.* $\diamond$

Proof

$$\begin{pmatrix} -E_c & -I_c \\ I_c & O_c \end{pmatrix} \cdot \begin{pmatrix} O_c & I_c \\ -I_c & -E_c \end{pmatrix} = \begin{pmatrix} I_c & O_c \\ O_c & I_c \end{pmatrix}$$

□

A consequence of the invertibility of W_c is that the sequence of its powers is periodic, as we will see. It is also useful to relate this to the (multiplicative) *order* $\operatorname{ord}(W_c)$ of W_c, which is by definition the smallest positive integer k such that $W_c^k = I_c$.

Theorem 4 *The sequence $(W_c^r)_{r \in \mathbb{N}}$ is purely periodic.* $\diamond$

Proof There are only finitely many different square matrices of size $2c$ over $\mathbb{Z}/n\mathbb{Z}$. So the sequence $(W_c^r)_{r \in \mathbb{N}}$ must become periodic. By the preceding lemma W_c is invertible so the sequence is periodic from the start. If the period starts with W_c, it must end with I_c; on the other hand, if $W_c^j = I_c$ for some $j > 0$ but smaller than the period length, then the sequence would repeat already from there on, a contradiction with the definition of period. □

As mentioned in [Lea17] there is a relation between the images of chasing the lights and Fibonacci polynomials. We find that relation reflected in our Structure Lemma. Note that T_r in this Lemma coincides with $T_{c,r}$ in Corollary 2.

Lemma 5 (Structure Lemma) *For all $c \in \mathbb{N}$ and for all $r \in \mathbb{N}$:*

$$W_c^r = \begin{pmatrix} T_r & -T_{r-1} \\ T_{r-1} & -T_{r-2} \end{pmatrix},$$

where the $c \times c$-matrices T_j over $\mathbb{Z}/n\mathbb{Z}$ are defined for $j \geq -1$ by the recursion $T_{-1} = O_c$, $T_0 = I_c$ and $T_{j+1} := -T_j \cdot E_c - T_{j-1}$. ◇

Proof By definition, $W_c = \left(\begin{smallmatrix} -E_c & -I_c \\ I_c & O_c \end{smallmatrix}\right) = \left(\begin{smallmatrix} T_1 & -T_0 \\ T_0 & -T_{-1} \end{smallmatrix}\right)$, as required.

The result now follows by induction on the exponent j:

$$W_c^{j+1} = W_c^j \cdot W_c = \begin{pmatrix} T_j & -T_{j-1} \\ T_{j-1} & -T_{j-2} \end{pmatrix} \cdot \begin{pmatrix} -E_c & -I_c \\ I_c & O_c \end{pmatrix} =$$
$$= \begin{pmatrix} -T_j \cdot E_c - T_{j-1} & -T_j \\ -T_{j-1} \cdot E_c - T_{j-2} & -T_{j-1} \end{pmatrix} \begin{pmatrix} T_{j+1} & -T_j \\ T_j & -T_{j-1} \end{pmatrix}.$$

□

3 Prime colours

With the Structure Lemma under our belt we are in a position to prove the observations we started out with. In fact, we can prove these observations in a generalized setting, but only for the case where the number of colours n is a prime number. The case of a composite number of colours requires somewhat more care, and is treated in the next section.

So, throughout this section we will assume that the number of colours n is prime; the importance of this is that then $\mathbb{Z}/n\mathbb{Z}$ is a finite field of n elements, which we will denote by $\mathbb{F}_n$. In this case we can reformulate Corollary 2.

Corollary 6 *For a prime number n the number of solvable initial configurations of $\mathcal{L}(r, c, n)$ will be $n^{r \cdot c - d}$, each allowing n^d different solutions, where $d = d(r, c)$ is* $\dim \operatorname{Ker} T_{c,r}$. ◇

The kernel dimensions $d(r, c)$ appeared in Table 1 for $n = 2$. Table 3 similarly lists these dimensions for Lights Out with $n = 3$ colours, for small values of r, c.

Proposition 7 (Observation 1) *Sequence $(d_c(r))_{r \in \mathbb{N}}$ is purely periodic.* ◇

Proof By the Structure Lemma, Corollary 6, the number of different solutions is determined by $\dim \operatorname{Ker} T_{c,r}$. Since $(W_c^r)_{r \in \mathbb{N}}$ is purely periodic, so is $(T_{c,r})_{r \in \mathbb{N}}$. □

$r \setminus c$:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	ℓ
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	3
2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	6
3	0	0	1	0	2	1	0	0	1	2	0	1	0	0	3	0	15
4	0	0	0	2	2	0	0	2	0	2	0	2	0	0	2	2	20
5	0	1	2	1	0	3	0	1	4	1	0	3	0	1	2	1	18
6	0	0	0	0	0	0	0	0	0	0	0	0	3	3	0	0	182
7	0	0	1	0	2	1	0	0	1	2	0	1	0	0	3	0	120
8	0	1	1	1	0	4	0	1	4	1	0	4	0	1	1	1	18
9	0	0	1	2	2	1	0	2	1	2	0	3	0	0	3	2	2460
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	122
11	0	1	2	1	2	3	0	1	4	3	0	3	0	1	8	1	90
12	0	0	0	0	0	0	3	0	0	0	0	0	6	6	0	0	182
13	0	0	1	0	0	1	3	0	1	0	0	1	6	6	1	0	546
14	0	1	1	3	2	2	0	3	1	3	0	8	0	1	7	3	60
15	0	0	1	0	2	1	0	0	1	2	0	1	0	0	3	0	9840
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	672605
17	0	1	2	1	0	5	0	1	8	1	0	5	0	1	2	1	54
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5097638
19	0	0	1	2	4	1	0	2	1	4	0	3	0	0	5	2	2460
20	0	1	1	1	0	2	0	1	1	1	0	2	3	4	1	1	546
21	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	44286
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	88573
23	0	1	2	1	2	3	0	1	4	3	0	3	0	1	8	1	360
24	0	0	0	2	2	0	0	2	0	2	0	2	0	0	2	2	174339220
25	0	0	1	0	0	1	3	0	1	0	0	1	6	6	1	0	546
26	0	1	1	1	0	4	0	1	4	1	0	4	0	1	1	1	54
27	0	0	1	0	2	1	3	0	1	2	0	1	6	6	3	0	199290
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5719198113740
29	0	1	2	3	2	3	0	3	4	3	0	9	0	1	8	3	7380
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	51472783023662
31	0	0	1	0	2	1	0	0	1	2	0	1	0	0	3	0	64570080
32	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	366

Table 2: Dimension of kernels, and length of period ℓ for $n = 3$

Note that the Proposition does not mention the *size* of either the period of the sequence $(W_c^r)_{r\in\mathbb{N}}$ or that of $(T_r)_{r\in\mathbb{N}}$ and hence of $(d_c(r))_{r\in\mathbb{N}}$ yet. But it is clear that the period of $(d_c(r))_{r\in\mathbb{N}}$ will be a divisor of the period of $(W_c^r)_{r\in\mathbb{N}}$. We will clarify the situation shortly.

Next on our agenda is our Observation 2, i.e., for every number of rows there will be a kernel of maximal dimension.

Lemma 8 (Observation 2) *For any given number of columns c there will be be a number of rows r such that $d_c(r) = c$.* ◇

Proof We will be using the notation as defined by the Structure Lemma.

There exists $p \in \mathbb{N}$ such that $W_c^p = I$. Then $W_c^{p-1} = W_c^{-1} \cdot W_c^p = W_c^{-1}$. Hence,

$$d_c(p) = \dim \operatorname{Ker} T_{-1} = \dim \operatorname{Ker} O_c = c.$$

□

By r_0 we will denote be the smallest positive r for which the maximal dimension of the kernel occurs. Before we investigate the period lengths further, we will take a closer look at Observation 4.

Theorem 9 (Observation 4) *For all $r \in \mathbb{N}$ the following inequality holds:*

$$d_c(r) + d_c(r+1) \leq c.$$

◇

Proof Assume, to the contrary, that $d_c(k) + d_c(k+1) > c$ for some $k \in \mathbb{N}$.

This means that there exist $m = d_c(k)$ independent press vectors in the kernel of T_k and $n = d_c(k+1)$ independent press vectors in the kernel of T_{k+1}. Since $m + n > c$ there must be a non-trivial press vector p in the intersection of both kernels. Then $(p \oplus o) \cdot W_c^k = o \oplus w$ for some vector w, and $(p \oplus o) \cdot W_c^{k+1} = o \oplus w'$ for some w'. But

$$o \oplus w' = (p \oplus o) \cdot W_c^{k+1} = (p \oplus o) \cdot W_c^k \cdot W_c = (o \oplus w) \cdot W_c = w \oplus o$$

and thus $w = o = w'$ and $p \oplus o$ is a non-trivial vector in the kernel of W_c^k, which contradicts the invertibility of W_c. □

Corollary 10 *If for some $r \geq 1$ we have $d_c(r) = c$ then*

- $d_c(r-1) = d_c(r+1) = 0$, *and*

- $W_c^{r+1} = \begin{pmatrix} -T_{r-1} & O_c \\ O_c & -T_{r-1} \end{pmatrix}$.

In particular: $d_c(r_0 - 1) = d_c(r_0 + 1) = 0$. ◇

Proof The first part is immediate by Theorem 9.

For the second statement, observe that $d_c(r) = c$ can only happen when $T_r = O_c$, so

$$W_c^{r+1} = W_c^r \cdot W_c = \begin{pmatrix} O_c & -T_{r-1} \\ T_{r-1} & -T_{r-2} \end{pmatrix} \cdot \begin{pmatrix} -E_c & -I_c \\ I_c & O_c \end{pmatrix} = \begin{pmatrix} -T_{r-1} & O_c \\ O_c & -T_{r-1} \end{pmatrix}.$$

The final statement follows from the definition of r_0. □

Corollary 11 *The period of sequence* $(d_c(r))_{r\in\mathbb{N}}$ *is* $r_0 + 1$. ◇

Proof We have already seen that the sequence is purely periodic; by definition $d_c(r) = \dim\ker T_r$ for $r \geq 0$. Here T_r is the upper left $c \times c$ submatrix of W_c^r. We also saw that $(W_c^r)_{r\in\mathbb{N}}$ forms a purely periodic sequence in Theorem 4, that can in fact be extended to a *bi-infinite* purely periodic sequence $(W_c^r)_{r\in\mathbb{Z}}$, since W_c is invertible by Lemma 3. This determines the bi-infinite sequence $(T_r)_{r\in\mathbb{Z}}$, for which we have a degree two recursion given in Lemma 5; this implies that the minimal period for $(d_c(r))_{r\in\mathbb{N}}$ is reached whenever two consecutive values repeat. Now $d_c(-1) = c$ as can be seen from the inverse of W_c in Lemma 3, so $d_c(-1) = d_c(r_0) = c$ and $d_c(0) = d_c(r_0 + 1) = 0$, and since $r_0 + 1$ is the smallest positive integer for which $d_c(r) = 0$, the period must be equal to $r_0 + 1$. □

Theorem 12 (Observation 4) *The period of* $(d_c(r))_{r\in\mathbb{N}}$ *ends as soon as the maximal dimension appears.* ◇

Note that conjugating any $2c \times 2c$ matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ by $Z = \left(\begin{smallmatrix} O_c & I_c \\ I_c & O_c \end{smallmatrix}\right)$ corresponds to rotating the four main sub-matrices:

$$Z \cdot M \cdot Z^{-1} = \begin{pmatrix} O_c & I_c \\ I_c & O_c \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} O_c & I_c \\ I_c & O_c \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}. \qquad (1)$$

This fact will be the linchpin in the proof of Observation 5.

But first we will see that a power of W_c and its inverse are conjugate.

Lemma 13 W_c^r *and* W_c^{-r} *are conjugate, for all* $r \in \mathbb{Z}$. ◇

Proof Omitting the subscripts c, we conjugate W by Z:

$$Z \cdot W \cdot Z^{-1} = \begin{pmatrix} O & I \\ I & O \end{pmatrix} \cdot \begin{pmatrix} -E & -I \\ I & O \end{pmatrix} \cdot \begin{pmatrix} O & I \\ I & O \end{pmatrix} = \begin{pmatrix} O & I \\ -I & -E \end{pmatrix} = W^{-1}.$$

from which the result follows immediately by taking r-th powers. □

Lemma 14 *If $W_c^r = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}$ for some $\in \mathbb{N}$ and some $c \times c$ matrix Q, then for all $i \in \mathbb{N}$:*

$$W_c^{r+i} = \begin{pmatrix} QT_i & -QT_{i-1} \\ QT_{i-1} & -QT_{i-2} \end{pmatrix} \quad \textit{and} \quad W_c^{r-i} = \begin{pmatrix} -QT_{i-2} & QT_{i-1} \\ -QT_{i-1} & QT_i \end{pmatrix}.$$

◇

Proof Note that $Z \cdot W_c^i \cdot Z^{-1} = W_c^{-i}$ for all i by the previous lemma, and that $Z \cdot W_c^r \cdot Z^{-1} = W_c^r$ if $W_c^r = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}$. Hence

$$Z \cdot W_c^{r+i} \cdot Z^{-1} = Z \cdot W_c^r \cdot Z^{-1} \cdot Z \cdot W^i \cdot Z^{-1} = W_c^r \cdot W_c^{-i} = W_c^{r-i}.$$

By direct calculation and the Structure Lemma we find

$$W_c^{r+i} = W_c^r W_c^i = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix} \begin{pmatrix} T_i & -T_{i-1} \\ T_{i-1} & -T_{i-2} \end{pmatrix} = \begin{pmatrix} QT_i & -QT_{i-1} \\ QT_{i-1} & -QT_{i-2} \end{pmatrix},$$

which, combined with Equation 1, yields what we set out to prove. □

Recall from the Structure Lemma the notation T_r for upper left $c \times c$ submatrix of W_c^r.

Corollary 15 *If $W_c^r = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}$ then for every i:*

$$T_{r-1+i} = T_{r-1-i} \quad \textit{and} \quad d_c(r-1+i) = d_c(r-1-i).$$

In particular, $d_c(r_0 + i) = d_c(r_0 - i)$. ◇

Proof By the preceding lemma we see that the upper left $c \times c$ submatrix of W^{r+i}, which is by definition T_{r+i}, equals the lower right $c \times c$ submatrix of W^{r-i}, which equals T_{r-i-2} by the Structure Lemma, and which is the upper left submatrix of W^{r-i-2} by definition. Hence $T_{r-1+(i+1)} = T_{r-1-(i+1)}$ for all i for which both sides are defined. The second statement follows as $d_c(k)$ is the dimension of the kernel of T_k. The final statement follows from Corollary 10 and taking $r = r_0 + 1$. □

Theorem 16 (Observation 5) *The the period of the sequence* $(d_c(r))_{r\in\mathbb{N}}$ *is almost palindromic: the period consist of a palindrome followed by the value c.* ◇

Proof By the last part of Corollary 15 the values preceding $d_c(r_0) = c$ which form the final part of the first period, mirror the values following it. □

Finally we clear up the relation between the order of W and the period length $r_0 + 1$ of d_c.

Lemma 17 *Either* $\operatorname{ord} W_c = r_0 + 1$ *or* $\operatorname{ord} W_c = 2(r_0 + 1)$. ◇

Proof Let us write $p = \operatorname{ord} W_c$ and $q = r_0 + 1$ in this proof.

By the definition of d_c its period length divides the order of W_c.

By Corollary 10 we know that $W^q = \left(\begin{smallmatrix} Q & O \\ O & Q \end{smallmatrix}\right)$ for a certain matrix Q, and by Lemma 14 $W_c^{2q} = W_c^0 = I_{2c}$. Then

$$I_{2c} = W_c^{2q} = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}^2 = \begin{pmatrix} Q^2 & O \\ O & Q^2 \end{pmatrix}$$

so $Q^2 = I_c$ and the order p of W equals q if $Q = I_c$ and equals $2q$ otherwise. □

4 Composite colours

In this section we will assume that the number of colours n will be a composite number. Trouble is then caused by the fact that $\mathbb{Z}/n\mathbb{Z}$ is a commutative ring that contains zero divisors, so it will not be field. That means that the linear algebra we want to do is not taking place in a vector space, but rather in the module $(\mathbb{Z}/n\mathbb{Z})^c$.

The results (and proofs) from Section 2 still hold: the matrix $W_c \in \mathcal{M}_{r,c}(\mathbb{Z}/n\mathbb{Z})$ is invertible, and its powers form a purely periodic sequence, the four blocks of which are given by the simple recurrence given by the Structure Lemma.

The complications arise when we try to explicitly compute the numbers of solvable configurations (and the number of different solutions in each case) as given by Corollary 2. This is due to the extra care the notions of 'independence' and 'basis' require in this case: it is, for example, not true that in any set of dependent vectors one one of them will be a linear combination of the others (in particular, 2 dependent vectors are not necessarily multiples of each other), and neither is it true that vectors in $(\mathbb{Z}/n\mathbb{Z})^c$ can only span subspaces of n^k vectors, for some k with $0 \leq k \leq c$.

What is still true, however, is the following alternative for the Dimension Theorem, which implies that for a $k \times k$ matrix M over a field the sum of the dimension of the kernel of M and its (row) rank are k: the number of elements K_M in $(\mathbb{Z}/n\mathbb{Z})^c$ that are in the kernel of M and the number of elements R_M $(\mathbb{Z}/n\mathbb{Z})^c$ that are in the span of the rows of M are related by $K_M \cdot R_M = 2^c$. This holds (as in the case where $\mathbb{Z}/n\mathbb{Z}$ is field) because the equivalence relation $x\tilde{y} \iff xM = yM$ partitions $(\mathbb{Z}/n\mathbb{Z})^c$ into cosets of size M of elements with the same image.

As a consequence, the observations from the Introduction do not generally hold in this case. Nonetheless, the following still holds.

Theorem 18 *For a fixed number of colours n and a fixed number of columns c, the number of solvable boards in $\mathcal{L}(n, c, r)$ forms a purely periodic sequence (as a function of the number of rows r), and so does the number of different solutions for each of these solvable boards. The common period of these sequences is a divisor of the period of the sequence of powers W_c^r of $W_c \in \mathcal{M}_{r,c}(\mathbb{Z}/n\mathbb{Z})$.* ◇

Proof This is an immediate consequence of Corollary 2, Theorem 4, and Lemma 5 and the remarks above. □

Example 19 In Tables 4 and 4 we list a small part of the sequences over $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. Let us consider the entry for $c = 5$ in the latter table in more detail.

The matrix W_5 has order 72 in this case (as the rightmost entry in row $r = 5$ in Table 4 indicates). For an $r \times 5$ board, the information we are looking for is provided by the 5×5 matrix T_r, which is the upper left submatrix of W_5^r. If we computer the numbers of elements of the kernel of T_r as a function of r, for $0 \leq r \leq 71$, we find the following sequence (the initial segment of which corresponds with the column $c = 5$ of the table): 1, 6, 18, 24, 1, 108, 1, 48, 162, 6, 1, 432, 1, 6, 18, 48, 1, 972, 1, 24, 18, 6, 1, 864, 1, 6, 162, 24, 1, 108, 1, 48, 18, 6, 1, 3888, 1, 6, 18, 48, 1, 108, 1, 24, 162, 6, 1, 864, 1, 6, 18, 24, 1, 972, 1, 48, 18, 6, 1, 432, 1, 6, 162, 48, 1, 108, 1, 24, 18, 6, 1, 7776. Note that,indeed,for $r = 71$ we find 6^5 elements in the kernel for the first time: T_{71} is the zero-matrix.

We take a closer look at the case $r = 5$, so the 5×5 board. The full

$r \setminus c$:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	ℓ
1	1	1	2^2	1	1	2^2	1	1	2^2	1	1	2^2	1	1	2^2	3
2	1	2^2	1	2^3	1	2^2	1	2^4	1	2^2	1	2^3	1	2^2	1	8
3	1	1	2^3	1	1	2^6	1	1	2^3	1	1	2^6	1	1	2^3	6
4	1	1	1	1	2^6	1	1	1	1	2^8	1	1	1	1	2^6	10
5	1	2^2	2^2	2^6	1	2^4	1	2^7	2^2	2^2	1	2^8	1	2^2	2^2	48
6	1	1	1	1	1	1	1	1	2^9	1	1	1	1	1	1	18
7	1	1	2^4	1	1	2^7	1	1	2^4	1	1	2^{13}	1	1	2^4	24
8	1	2^2	1	2^3	1	2^2	2^9	2^4	1	2^2	1	2^3	1	2^{14}	1	56
9	1	1	2^2	1	2^8	2^2	1	1	2^2	2^{12}	1	2^2	1	1	2^{10}	60
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	62
11	1	2^2	2^3	2^6	1	2^8	1	2^{13}	2^3	2^2	1	2^{12}	1	2^2	2^3	96
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	126
13	1	1	2^2	1	1	2^2	1	1	2^{14}	1	1	2^2	1	1	2^2	36
14	1	2^2	1	2^3	2^6	2^2	1	2^4	1	2^{10}	1	2^3	1	2^2	2^6	680
15	1	1	2^4	1	1	2^8	1	1	2^4	1	1	2^{15}	1	1	2^4	48
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2^{12}	510
17	1	2^2	2^2	2^6	1	2^4	2^{12}	2^7	2^2	2^2	1	2^8	1	2^{20}	2^2	336
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1026
19	1	1	2^3	1	2^8	2^6	1	1	2^3	2^{16}	1	2^6	1	1	2^{11}	120
20	1	2^2	1	2^3	1	2^2	1	2^4	2^9	2^2	1	2^3	1	2^2	1	4680
21	1	1	2^2	1	1	2^2	1	1	2^2	1	1	2^2	1	1	2^2	372
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4094
23	1	2^2	2^4	2^6	1	2^9	1	2^{14}	2^4	2^2	1	2^{19}	1	2^2	2^4	192
24	1	1	1	1	2^6	1	1	1	1	2^8	1	1	1	1	2^6	2050
25	1	1	2^2	1	1	2^2	1	1	2^2	1	1	2^2	1	1	2^2	252
26	1	2^2	1	2^3	1	2^2	2^9	2^4	1	2^2	1	2^3	1	2^{14}	1	4088
27	1	1	2^3	1	1	2^6	1	1	2^{15}	1	1	2^6	1	1	2^3	72
28	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	6554
29	1	2^2	2^2	2^6	2^8	2^4	1	2^7	2^2	2^{14}	1	2^8	1	2^2	2^{10}	4080
30	1	1	1	1	1	1	1	1	1	1	2^{15}	1	1	1	1	682
31	1	1	2^4	1	1	2^8	1	1	2^4	1	1	2^{16}	1	1	2^4	96
32	1	2^2	1	2^3	1	2^2	1	2^4	1	2^2	1	2^3	1	2^2	1	8184

Table 3: Size of kernels and period ℓ for $n = 4$

$r \setminus c$:	0	1	2	3	4	5	6	7	8	9	10	ℓ
0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	$2 \cdot 3$	1	1	$2 \cdot 3$	1	1	$2 \cdot 3$	1	1	3
2	1	$2 \cdot 3$	3	$2^2 \cdot 3$	1	$2 \cdot 3^2$	1	$2^2 \cdot 3$	3	$2 \cdot 3$	1	12
3	1	1	$2^2 \cdot 3$	1	3^2	$2^3 \cdot 3$	1	1	$2^2 \cdot 3$	3^2	1	30
4	1	1	1	3^2	$2^4 \cdot 3^2$	1	1	3^2	1	$2^4 \cdot 3^2$	1	20
5	1	$2 \cdot 3$	$2 \cdot 3^2$	$2^3 \cdot 3$	1	$2^2 \cdot 3^3$	1	$2^4 \cdot 3$	$2 \cdot 3^4$	$2 \cdot 3$	1	72
6	1	1	1	1	1	1	1	1	2^6	1	1	1638
7	1	1	$2^2 \cdot 3$	1	3^2	$2^4 \cdot 3$	1	1	$2^2 \cdot 3$	3^2	1	120
8	1	$2 \cdot 3$	3	$2^2 \cdot 3$	1	$2 \cdot 3^4$	2^6	$2^2 \cdot 3$	3^4	$2 \cdot 3$	1	252
9	1	1	$2 \cdot 3$	3^2	$2^4 \cdot 3^2$	$2 \cdot 3$	1	3^2	$2 \cdot 3$	$2^8 \cdot 3^2$	1	2460
10	1	1	1	1	1	1	1	1	1	1	1	3782
11	1	$2 \cdot 3$	$2^2 \cdot 3^2$	$2^3 \cdot 3$	3^2	$2^4 \cdot 3^3$	1	$2^7 \cdot 3$	$2^2 \cdot 3^4$	$2 \cdot 3^3$	1	720
12	1	1	1	1	1	1	3^3	1	1	1	1	1638
13	1	1	$2 \cdot 3$	1	1	$2 \cdot 3$	3^3	1	$2^7 \cdot 3$	1	1	1638
14	1	$2 \cdot 3$	3	$2^2 \cdot 3^3$	$2^4 \cdot 3^2$	$2 \cdot 3^2$	1	$2^2 \cdot 3^3$	3	$2^5 \cdot 3^3$	1	1020
15	1	1	$2^2 \cdot 3$	1	3^2	$2^4 \cdot 3$	1	1	$2^2 \cdot 3$	3^2	1	9840
16	1	1	1	1	1	1	1	1	1	1	1	2017815
17	1	$2 \cdot 3$	$2 \cdot 3^2$	$2^3 \cdot 3$	1	$2^2 \cdot 3^5$	2^6	$2^4 \cdot 3$	$2 \cdot 3^8$	$2 \cdot 3$	1	1512
18	1	1	1	1	1	1	1	1	1	1	1	2615088294
19	1	1	$2^2 \cdot 3$	3^2	$2^4 \cdot 3^4$	$2^3 \cdot 3$	1	3^2	$2^2 \cdot 3$	$2^8 \cdot 3^4$	1	2460
20	1	$2 \cdot 3$	3	$2^2 \cdot 3$	1	$2 \cdot 3^2$	1	$2^2 \cdot 3$	$2^6 \cdot 3$	$2 \cdot 3$	1	16380
21	1	1	$2 \cdot 3$	1	1	$2 \cdot 3$	1	1	$2 \cdot 3$	1	1	1372866
22	1	1	1	1	1	1	1	1	1	1	1	7882997
23	1	$2 \cdot 3$	$2^2 \cdot 3^2$	$2^3 \cdot 3$	3^2	$2^5 \cdot 3^3$	1	$2^7 \cdot 3$	$2^2 \cdot 3^4$	$2 \cdot 3^3$	1	1440
24	1	1	1	3^2	$2^4 \cdot 3^2$	1	1	3^2	1	$2^4 \cdot 3^2$	1	35739540100
25	1	1	$2 \cdot 3$	1	1	$2 \cdot 3$	3^3	1	$2 \cdot 3$	1	1	1638
26	1	$2 \cdot 3$	3	$2^2 \cdot 3$	1	$2 \cdot 3^4$	2^6	$2^2 \cdot 3$	3^4	$2 \cdot 3$	1	55188
27	1	1	$2^2 \cdot 3$	1	3^2	$2^3 \cdot 3$	3^3	1	$2^8 \cdot 3$	3^2	1	1195740
28	1	1	1	1	1	1	1	1	1	1	1	646269386852620
29	1	$2 \cdot 3$	$2 \cdot 3^2$	$2^3 \cdot 3^3$	$2^4 \cdot 3^2$	$2^2 \cdot 3^3$	1	$2^4 \cdot 3^3$	$2 \cdot 3^4$	$2^9 \cdot 3^3$	1	250920
30	1	1	1	1	1	1	1	1	1	1	2^{10}	51472783023662
31	1	1	$2^2 \cdot 3$	1	3^2	$2^4 \cdot 3$	1	1	$2^2 \cdot 3$	3^2	1	64570080
32	1	$2 \cdot 3$	3	$2^2 \cdot 3$	1	$2 \cdot 3^2$	1	$2^2 \cdot 3$	3	$2 \cdot 3$	1	249612

Table 4: Size of kernels and period ℓ for $n = 6$

matrix W_5^5, of which T_5 is the upper left block) equals

$$W_5^5 = W_c := \begin{pmatrix} 4 & 5 & 5 & 2 & 1 & 4 & 0 & 0 & 4 & 1 \\ 5 & 3 & 1 & 0 & 2 & 0 & 4 & 4 & 1 & 4 \\ 5 & 1 & 4 & 1 & 5 & 0 & 4 & 5 & 4 & 0 \\ 2 & 0 & 1 & 3 & 5 & 4 & 1 & 4 & 4 & 0 \\ 1 & 2 & 5 & 5 & 4 & 1 & 4 & 0 & 0 & 4 \\ 2 & 0 & 0 & 2 & 5 & 2 & 3 & 3 & 1 & 0 \\ 0 & 2 & 2 & 5 & 2 & 3 & 5 & 4 & 3 & 1 \\ 0 & 2 & 1 & 2 & 0 & 3 & 4 & 5 & 4 & 3 \\ 2 & 5 & 2 & 2 & 0 & 1 & 3 & 4 & 5 & 3 \\ 5 & 2 & 0 & 0 & 2 & 0 & 1 & 3 & 3 & 2 \end{pmatrix}$$

and a small computation shows that the kernel of T_5 is generated by the following 3 independent vectors over $\mathbb{Z}/6\mathbb{Z}$:

$$(1,0,1,0,3),(0,1,1,5,4),(0,0,2,0,2).$$

Note that the third of these only spans a submodule of size 3, whence the kernel consists of $6^2 \cdot 3$ instead of 6^3 elements. This vector correpsonds to the following pressing pattern in the kernel:

$$W_5^5 = W_c := \begin{pmatrix} 0 & 0 & 2 & 0 & 2 \\ 0 & 4 & 4 & 2 & 4 \\ 2 & 4 & 0 & 2 & 4 \\ 0 & 2 & 2 & 4 & 2 \\ 2 & 4 & 4 & 2 & 2 \\ , & & & & \end{pmatrix}$$

The total number of light patterns that can be created from a totally unlit board (or equivalently, the number of different light patterns that can be turned off completely) equals $\frac{6^{25}}{6^2 \cdot 3} =$. ▷

5 Periods

Finally, in this section, we explain how the periods in our tables were computed. Neither computing the powers of W_c directly, nor using the recursion for $T_{c,r}$ will be sufficiently efficient to determine the periods we want; however, using Theorem 4 and Lemma 17 we will be able to determine the length of the period (but not the values in it) if we can compute the order of the matrix W_c. We make use of the fact that W_c is a symplectic matrix.

Definition For a commutative ring R and a positive integer c, the *symplectic group* $\mathrm{Sp}(2c, R)$ is the multiplicative group of $(2c) \times (2c)$ matrices M over R satisfying

$$M \cdot J \cdot M^{\mathrm{T}} = J,$$

where J is the matrix with four $c \times c$ blocks

$$J = \begin{pmatrix} O_c & I_c \\ -I_c & O_c \end{pmatrix}.$$

○

The order of $\mathrm{Sp}(2c, \mathbb{F}_p)$ can be found in many textbooks; for symplectic matrices over $\mathbb{Z}/n\mathbb{Z}$ generally, it took some effort to find a reference to the following formula. It appears in [?], p. 136.

Theorem 20 *The order of the group of* $2c \times 2c$ *symplectic matrices over* $\mathbb{Z}/n\mathbb{Z}$ *equals*

$$G(c, n) = n^{2c^2+c} \cdot \prod_{\substack{p|c \\ p \text{ prime}}} \cdot \prod_{k=1}^{c} \left(1 - \frac{1}{p^{2k}}\right).$$

◇

Clearly, the order of W_c over $\mathbb{Z}/n\mathbb{Z}$ is a divisor of the group order given by the theorem. The group order $G(c, n)$ grows quickly with c, and the order of W_c is usually much smaller. Fortunately, the group order is highly composite, as a product of reasonably small primes. From the prime factorization of the group order $G(c, n)$ we compute the order of W_c by a standard technique: for each maximal prime power p^m of $G(c, n)$ determine the true power of p dividing the order of W_c by using (fast) p-powering of W_c to compute the powers $1, p, p^2, \ldots p^m$ of $W_c^{G(c,n)/p^m}$ until the result is the identity matrix.

Example 21 As an example, let us look at $n = 6$ and $c = 16$; then $G(c, n)$ is a 411 decimal digit number with prime factorization

$2^{319} \cdot 3^{278} \cdot 5^{18} \cdot 7^{10} \cdot 11^9 \cdot 13^7 \cdot 17^6 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31^4 \cdot 37 \cdot 41^5 \cdot 43^2 \cdot 61^3 \cdot 67 \cdot 73^3 \cdot 89 \cdot$

$113 \cdot 127^2 \cdot 151 \cdot 193^2 \cdot 241 \cdot 257^2 \cdot 271 \cdot 331 \cdot 547^2 \cdot 661 \cdot 683 \cdot 757 \cdot 1093^2 \cdot 1181 \cdot$

$2731 \cdot 3851 \cdot 4561 \cdot 6481 \cdot 8191 \cdot 16493 \cdot 65537 \cdot 398581 \cdot 797161 \cdot 21523361,$

while it turns out that the order of W_{16} is

$$4035630 = 2 \cdot 3 \cdot 5 \cdot 17 \cdot 41 \cdot 193.$$

$r \setminus c$:	0	1	2	3	4	5	6	7	8	9	10	ℓ
0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	$2\cdot3\cdot5$	1	1	$2\cdot3\cdot5$	1	1	$2\cdot3\cdot5$	1	1	3
2	1	$2\cdot3\cdot5$	3	$2^2\cdot3\cdot5$	5	$2\cdot3^2\cdot5$	1	$2^2\cdot3\cdot5$	3	$2\cdot3\cdot5^2$	1	60
3	1	1	$2^2\cdot3\cdot5$	1	3^2	$2^3\cdot3\cdot5$	1	1	$2^2\cdot3\cdot5$	3^2	1	390
4	1	1	5	3^2	$2^4\cdot3^2\cdot5^2$	5	1	3^2	5	$2^4\cdot3^2\cdot5^2$	1	60
5	1	$2\cdot3\cdot5$	$2\cdot3^2\cdot5$	$2^3\cdot3\cdot5$	5	$2^2\cdot3^3\cdot5^2$	1	$2^4\cdot3\cdot5$	$2\cdot3^4\cdot5$	$2\cdot3\cdot5^2$	1	4680
6	1	1	1	1	1	1	1	1	2^6	1	1	50778
7	1	1	$2^2\cdot3\cdot5$	1	3^2	$2^4\cdot3\cdot5$	1	1	$2^2\cdot3\cdot5$	3^2	1	488280
8	1	$2\cdot3\cdot5$	3	$2^2\cdot3\cdot5$	5	$2\cdot3^4\cdot5$	2^6	$2^2\cdot3\cdot5$	3^4	$2\cdot3\cdot5^2$	1	1260
9	1	1	$2\cdot3\cdot5^2$	3^2	$2^4\cdot3^2\cdot5^2$	$2\cdot3\cdot5^2$	1	3^2	$2\cdot3\cdot5^2$	$2^8\cdot3^2\cdot5^2$	1	2460
10	1	1	1	1	1	1	1	1	1	1	1	2953742
11	1	$2\cdot3\cdot5$	$2^2\cdot3^2\cdot5$	$2^3\cdot3\cdot5$	$3^2\cdot5$	$2^4\cdot3^3\cdot5^2$	1	$2^7\cdot3\cdot5$	$2^2\cdot3^4\cdot5$	$2\cdot3^3\cdot5^2$	1	9360
12	1	1	1	5^2	1	5^2	3^3	5^2	1	1	1	3276
13	1	1	$2\cdot3\cdot5$	1	1	$2\cdot3\cdot5$	3^3	1	$2^7\cdot3\cdot5$	1	1	50778
14	1	$2\cdot3\cdot5$	$3\cdot5$	$2^2\cdot3^3\cdot5$	$2^4\cdot3^2\cdot5^4$	$2\cdot3^2\cdot5^2$	1	$2^2\cdot3^3\cdot5$	$3\cdot5$	$2^5\cdot3^3\cdot5^9$	1	5100
15	1	1	$2^2\cdot3\cdot5$	1	3^2	$2^4\cdot3\cdot5$	1	1	$2^2\cdot3\cdot5$	3^2	1	7820129394480
16	1	1	1	1	1	1	1	1	1	1	1	188660621641830
17	1	$2\cdot3\cdot5$	$2\cdot3^2\cdot5$	$2^3\cdot3\cdot5$	5	$2^2\cdot3^5\cdot5^2$	2^6	$2^4\cdot3\cdot5$	$2\cdot3^8\cdot5$	$2\cdot3\cdot5^2$	1	3046680
18	1	1	1	1	1	1	1	1	1	1	1	67205154067506
19	1	1	$2^2\cdot3\cdot5^2$	3^2	$2^4\cdot3^4\cdot5^2$	$2^3\cdot3\cdot5^2$	1	3^2	$2^2\cdot3\cdot5^2$	$2^8\cdot3^4\cdot5^2$	1	31980
20	1	$2\cdot3\cdot5$	3	$2^2\cdot3\cdot5$	5	$2\cdot3^2\cdot5$	1	$2^2\cdot3\cdot5$	$2^6\cdot3$	$2\cdot3\cdot5^2$	1	507780
21	1	1	$2\cdot3\cdot5$	1	1	$2\cdot3\cdot5$	1	1	$2\cdot3\cdot5$	1	1	97473486
22	1	1	1	1	1	1	1	1	1	1	1	102144181728362994267
23	1	$2\cdot3\cdot5$	$2^2\cdot3^2\cdot5$	$2^3\cdot3\cdot5$	$3^2\cdot5$	$2^5\cdot3^3\cdot5^2$	1	$2^7\cdot3\cdot5$	$2^2\cdot3^4\cdot5$	$2\cdot3^3\cdot5^2$	1	5859360
24	1	1	5	3^2	$2^4\cdot3^2\cdot5^2$	5	1	3^2	5	$2^4\cdot3^2\cdot5^2$	1	107218620300
25	1	1	$2\cdot3\cdot5$	5^2	1	$2\cdot3\cdot5^3$	3^3	5^2	$2\cdot3\cdot5$	1	1	1025388
26	1	$2\cdot3\cdot5$	3	$2^2\cdot3\cdot5$	5	$2\cdot3^4\cdot5$	2^6	$2^2\cdot3\cdot5$	3^4	$2\cdot3\cdot5^2$	1	1425781980
27	1	1	$2^2\cdot3\cdot5$	1	3^2	$2^3\cdot3\cdot5$	3^3	1	$2^8\cdot3\cdot5$	3^2	1	37067940
28	1	1	1	1	1	1	1	1	1	1	1	5667407040183716027780
29	1	$2\cdot3\cdot5$	$2\cdot3^2\cdot5^2$	$2^3\cdot3^3\cdot5$	$2^4\cdot3^2\cdot5^4$	$2^2\cdot3^3\cdot5^3$	1	$2^4\cdot3^3\cdot5$	$2\cdot3^4\cdot5^2$	$2^9\cdot3^3\cdot5^9$	1	16309800
30	1	1	1	1	1	1	5^3	1	1	1	2^{10}	463255047212958
31	1	1	$2^2\cdot3\cdot5$	1	3^2	$2^4\cdot3\cdot5$	1	1	$2^2\cdot3\cdot5$	3^2	1	230298610404133796691884640
32	1	$2\cdot3\cdot5$	3	$2^2\cdot3\cdot5$	5	$2\cdot3^2\cdot5$	1	$2^2\cdot3\cdot5$	3	$2\cdot3\cdot5^2$	1	46166987460

Table 5: Size of kernels and period ℓ for $n = 30$

In this case it is just doable to compute the numbers that make up the full period (which turns out to be half the order of W_{16}; compare Lemma 17).

For $q = 6$ and $c = 18$ computing the complete period is not feasible in reasonable time, but the length is easily established. ▷

Finally, just to show that this is easily done, we give show Table 5, for 30 colours.

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